Student Name: ______________________________

Class: ______________________________

Date: ______________________________

Instructions: **Read each question carefully and select the correct answer.**

1. Integrate.

$$\int\left(18x^5+3x^2+3\right)dx$$

A. $90x^4 + 6x + C$
B. $3x^6+x^3+3x+C$
C. $\frac{18x^6}{5}+\frac{3x^2}{2}+3x+C$
D. $\frac{18x^6}{5}+\frac{3x^3}{2}+C$

2. Integrate.

$$\int x\left(3-2x^2\right)^3\,dx$$

A. $-\frac{\left(3-2x^2\right)^4}{16}+C$
B. $\frac{\left(3-2x^2\right)^4}{16}+C$
C. $\frac{x^2\left(3-2x^2\right)^4}{4}+C$
D. $-\frac{x^2\left(3-2x^2\right)^4}{4}+C$

3. Find the derivative of the following function using the product rule.

$$f(x)=\left(\frac{1}{2}x^2\right)\left(4x^3\right)$$

A. $f'(x) = 12x^3$
B. $f'(x) = 4x^4$
C. $f'(x) = 10x^4$
D. $f'(x) = 80x^4$

4. Find the derivative of the following function.

$$f(x) = 9x^3 - 3x^2 - 4$$

A. $f'(x)=\frac{9x^4}{4}-x^3-4x$
B. $f'(x) = 9x^2 - 3x$
C. $f'(x) = 27x^3 - 6x^2 - 4$
D. $f'(x) = 27x^2 - 6x$

5. Find the cross product A x B for A = <9, 4, -3> and B = <6, -9, 7>

A. <54, -36, -21>
B. <1, -81, -105>
C. -3
D. 9

6. Find -3A + 5B for A = <6, 1, -2> and B = <-4, 8, 3>.

A. 20
B. <-15, 35>
C. <2, -43, -9>
D. <-38, 37, 21>

7. Evaluate the limit.

$$\lim_{t \to 3} \frac{t^2 + t - 12}{t^2 - 9}$$

A. 0
B. 4/3
C. 7/6
D. The limit does not exist.

8. Evaluate the limit.

$$\lim_{t \to 0} \frac{(4 + t)^2 - 16}{t}$$

A. 0
B. 8
C. 16
D. The limit does not exist.

12th All Strands
Answer Key
03/12/2008

1.	**B**	Integrals
2.	**A**	Integrals
3.	**C**	Derivatives
4.	**D**	Derivatives
5.	**B**	Vectors
6.	**D**	Vectors
7.	**C**	Limits
8.	**B**	Limits

Study Guide

12th All Strands
03/12/2008

Integrals

Another term for indefinite integral is antiderivative. Taking the integral of a function is the opposite of taking the derivative.

The integral sign is the symbol $\int$.
The process of taking an integral is called antidifferentiation or integration. The function being integrated is called the integrand. There are rules for integrating functions. The following is a description of some of the rules and an example of the rule being used. You will notice that in each rule, C is added to the end of each function after it has been integrated. C is an arbitrary constant and is added to the integral because the integral is the reverse of the derivative and the derivative of a whole number is always zero. Since the derivative of a whole number is always zero, the zero is always dropped off the derivative. We need to add the C just in case there was a whole number added to the end of the original function.

The answer to an integration problem can be checked by taking the derivative of the answer. If that derivative matches the integrand, the problem was most likely done correctly.

Rule 1

$\int a \, dx = ax + c$

Where 'a' is a constant

The term dx is used in the rule to identify x as the variable of integration in the problem.

Example 1:

Find the integral of $f(x) = \int 3 \, dx$.

(1) a = 3
(2) 3x + C

Step 1: Since 3 is in the same place as the a in the rule, a = 3.
Step 2: Substitute a = 3 into the correct rule for integration.

The correct answer is f(x) = 3x + C.

Rule 2

$\int (ax + b) \, dx = \frac{a}{2} x^2 + bx + C$

Example 2: Integrate $f(x) = \int 3x + 6 \, dx$.

(1) $a = 3, \; b = 6$

(2) $\frac{3}{2} x^2 + 6x + C$

Step 1: Since 3 is in the same place as a and 6 is in the same place as b, a = 3 and b = 6.
Step 2: Substitute a = 3 and b = 6 into the correct rule for integration. Then simplify, if necessary.

The correct answer is $f(x) = \frac{3}{2} x^2 + 6x + C$.

Rule 3: The Power Rule

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$

Example 3: Integrate the following. Remember to distribute the integration to each term and then add or subtract the integrals.

$f(x) = \int 6x^5 + 3x^4 - 2x^3 \, dx$

$$(1)\ f(x) = \int 6x^5 dx + \int 3x^4 dx + \int -2x^3 dx$$

$$(2)\ \int 6x^5 dx = \frac{6x^{5+1}}{5+1} dx = \frac{6x^6}{6} = x^6$$

$$\int 3x^4 dx = \frac{3x^{4+1}}{4+1} = \frac{3x^5}{5}$$

$$\int -2x^3 dx = \frac{-2x^{3+1}}{3+1} = \frac{-2x^4}{4} = \frac{-x^4}{2}$$

$$(3)\ x^6 + \frac{3x^5}{5} - \frac{x^4}{2} + C$$

Step 1: The integral can be split into three separate integrals and each integral can then be performed separately. The 'dx' term needs to be included in each new integral to show that the variable to be integrated is 'x'.
Step 2: Take the integral of each function separately. In the first integral, n = 5 because the exponent on the x term is 5. In the second integral n = 4 and in the third integral n = 3. Substitute the value of 'n' into the correct rule for integration for each integrand, then simplify.
Step 3: Now that all three integrals have been taken, the three functions can be put back together as one long function. Remember to use the same operations as the original integrand.

The correct answer is
$f(x) = x^6 + \frac{3}{5}x^5 - \frac{1}{2}x^4 + C$.

Rule 4

$$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)} + C$$

Example 4: Integrate the following.

$$f(x) = \int \frac{dx}{(2x+3)^2}$$

$$(1)\ a = 3,\ b = 2$$

$$(2)\ \frac{{}^{-}1}{2(3+2x)} + C$$

Step 1: Since 2 is multiplied by the x in 2x + 3, b = 2. Three is the constant term (not multiplied by a variable) in 2x + 3, so a = 3.
Step 2: Substitute a = 3 and b = 2 into the correct rule for integration and simplify.

The correct answer is
$f(x) = \frac{1}{2(3+2x)} + C$.

Rule 5

$$\int (a+bx)^n dx = \frac{1}{(n+1)b}(a+bx)^{n+1} + C,\ n \neq {}^{-}1$$

Example 5: Find the integral of the following.

$$f(x) = \int \sqrt{2x+4}\ dx$$

$$(1)\ \int (2x+4)^{\frac{1}{2}}\ dx$$

$$(2)\ a = 4,\ b = 2,\ n = \frac{1}{2}$$

$$(3)\ \frac{1}{\left(\frac{1}{2}+1\right)2}(2x+4)^{\frac{1}{2}+1} + C$$

$$(4)\ \frac{1}{\left(\frac{3}{2}\right)2}(2x+4)^{\frac{3}{2}} + C$$

$$(5)\ \frac{1}{3}(2x+4)^{\frac{3}{2}} + C$$

Step 1: Rewrite the integral. The square root of (2x + 4) can be written as (2x + 4) to the one-half power. Now we can use rule 5 to determine the integral.
Step 2: Since 4 is the constant term in 2x + 4, a = 4. Two is multiplied by x in 2x + 4, so b = 2. The exponent on the function is 1/2, so n = 1/2.
Step 3: Substitute a = 4, b = 2, and n = 1/2 into the correct rule for integration.
Step 4: 1/2 + 1 can be written as 1/2 + 2/2. 1/2 + 2/2 = 3/2. Replace all 1/2 + 1 with 3/2.

Step 5: 3/2 x 2 can be written as 3/2 x 2/1. 3/2 x 2/1 = 6/2. 6/2 can be reduced to 3. Now the integral is simplified.

The correct answer is

$$f(x) = \frac{1}{3}(2x+4)^{\frac{3}{2}} + C.$$

Example 6: Integrate the following function.

$$f(x) = \int \frac{3}{x^4}\,dx$$

(1)	(2)	(3)	(4)	(5)	(6)
$\int 3x^{-4}dx$	$n = -4$	$\frac{3x^{-4+1}}{-4+1} + C$	$\frac{3x^{-3}}{-3} + C$	$-1x^{-3} + C$	

Step 1: Rewrite the function. x^4 Now we can use the Power Rule to integrate the function.
Step 2: The exponent on the x term is -4, so n = -4. Make sure that n does not equal -1, so the Power Rule can be used.
The antiderivative of x^{-4} will be multiplied by 3.
Step 3: Substitute n = -4 into the Power Rule.
Step 4: -4 + 1 = -3. Substitute -3 in place of all (-4 + 1).
Step 5: 3 (on top of the fraction) can be divided by -3. 3 ÷ -3 = -1
Step 6: Since it is not proper to have a negative exponent in the answer, we need to move the x^{-3} to the bottom of the fraction. Once the term is moved to the bottom, the sign of the exponent changes.

The correct answer is $f(x) = \frac{-1}{x^3} + C$.

Evaluating an Integral:

An indefinite integral is an integral for which there is an unknown value. The integrals above are examples of indefinite integrals. A definite integral is the result of integrating a function between two specified values of the variable. A definite integral can be evaluated using the Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus

If f is a continuous function on a clos

$[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Where F is the antiderivative of the fu

The first step in evaluating an integral is determining the antiderivative of the function. Then substitute the value of 'b' into the antiderivative and simplify. Next substitute the value of 'a' into the antiderivative and simplify. Finally, subtract the two calculated values. Notice that the constant term 'C' is not added to the end of the integrals when you are evaluating a definite integral. This is because there is no unknown value in a definite integral.

Example 7: Evaluate the integral.

$$\int_1^3 4x^5 - 3x^3 + 3\,dx$$

(1)

$$\int 4x^5dx = \frac{4x^{5+1}}{5+1} = \frac{4x^6}{6} = \frac{2x^6}{3}$$

$$\int 3x^3dx = \frac{3x^{3+1}}{3+1} = \frac{3x^4}{4}$$

$$\int 3\ dx = 3x$$

Step 1: Separate the terms of the function to be integrated and determine the integral of

each function. (Use the rules from the previous examples to complete the integration.)

(2)

$$\left.\frac{2x^6}{3} - \frac{3x^4}{4} + 3x\right]_1^3$$

Step 2: Now put the integrals back together as one long function. Place a large bracket at the end with the value of a at the bottom corner and the value of b at the top corner.

(3)

$$\left[\frac{2(3)^6}{3} - \frac{3(3)^4}{4} + 3(3)\right] - \left[\frac{2(1)^6}{3} - \frac{3(1)}{4} + 3(1)\right]$$

Step 3: We now use the Fundamental Theorem of Calculus to evaluate the integral. First, substitute the value b = 3 into the function and place brackets around the new function. Then substitute the value a = 1 into the function and place brackets around this new function. Now put a subtraction symbol in between the two sets of brackets.

(4)

$$\left[\frac{2(729)}{3} - \frac{3(81)}{4} + 3(3)\right] - \left[\frac{2(1)}{3} - \frac{3(1)}{4} + 3(1)\right]$$

Step 4: We can now begin to simplify the new functions. First, we simplify all numbers with exponents.

(5)

$$\left[\frac{1458}{3} - \frac{243}{4} + 9\right] - \left[\frac{2}{3} - \frac{3}{4} + 3\right]$$

Step 5: Now perform all multiplications.

(6)

$$[486 - 60.75 + 9] - [0.67 - 0.75 + 3]$$

Step 6: The next step is to perform all operations involving division. All terms should either be whole numbers or decimal numbers at this point.

(7)

$$434.25 - 2.92$$

Step 7: Determine the value in each set of brackets by adding and/or subtracting.

(8)

$$431.33$$

Step 8: Subtract 2.92 from 434.25.

The correct answer is 431.33.

Derivatives

A function f(x) is a relation between two variables such that each value of the first variable (x) corresponds to exactly one value of the second variable f(x). A derivative f'(x) of a function **f** depicts how the function **f** is changing at point **x**. It is necessary for the function **f** to be continuous at point x in order for there to be a derivative at that point. A continuous function is a function that has no "breaks", "holes", or "tears." A function which has a derivative is said to be differentiable.

Derivatives are denoted in many ways. Let's begin with the function y = f(x). The derivative can be denoted in the following

ways.

$$\frac{dy}{dx}, \frac{d}{dx}f(x), f', \text{ or } f'(x)$$

The symbol $\frac{dy}{dx}$ means "the derivative with respect to x of..."

Computing a Derivative:

The following are the rules for determining the derivative of a function and an example of each.

Rule 1 : Constant Rule

If f is a constant function $f(x) = c$, then $f'(x) = 0$

That is, $\frac{d}{dx}(c) = 0$.

A constant term is a number that does **not** have a variable. Constant terms are generally denoted with the letter 'c'. The constant rule simply states that the derivative of a constant term equals zero.

Example 1: Find the derivative of f(x) = 3.

Step 1: Since the number 3 is a constant, its derivative is 0.

The derivative of f(x) = 3 is f'(x) = 0.

Rule 2 : Power Rule

Let n be any real number with $n \neq 0$.

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

That is, $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ (provided $x \neq 0$

In this rule, the variable n denotes an exponent on the x variable. To take the derivative of a term with an exponent, multiply the variable by the value of the exponent and then subtract one from the exponent.

Example 2: Find the derivative of $f(x) = x^6$.

(1) $f'(x) = 6 \cdot x^{6-1}$

(2) $f'(x) = 6x^5$

Step 1: It can be determined that n = 6, because 6 is the exponent. Now, using the Power Rule, we substitute the 6 in place of the n in $f'(x) = nx^{n-1}$.

Step 2: Multiply the variable x by 6. Subtract 1 from 6 to get 5 which is the exponent on the x for the derivative.

The derivative is $f'(x) = 6x^5$.

Rule 3: Constant Multiple Rule

If f is a Function such that $f(x) = cx^n$,

then $f'(x) = c \cdot nx^{n-1}$.

That is, $\frac{d}{dx} cx^n = c \cdot \frac{d}{dx}\left(x^n \right) = c \cdot n \cdot x^{n-1}$

This rule states what should be done when there is a constant c multiplied by a variable x taken to a power n. First, the constant c would be multiplied by the exponent n. Then one would be subtracted from the exponent n.

Example 3: Find the derivative of $f(x) = 3x^7$.

(1) $f'(x) = 7 \cdot 3x^{7-1}$

(2) $f'(x) = 21x^6$

<u>Step 1</u>: In order to use the Constant Multiple Rule, we must know the value of c and the value of n. The constant in front of the x is 3, so c = 3. The exponent on x is 7, so n = 7. Substitute the values of c and n into the rule.
<u>Step 2</u>: Multiply 7 by 3 to get 21. Subtract 1 from 7 to get 6, the exponent on x for the derivative.

The derivative is $f'(x) = 21x^6$.

Rule 4: Sum Rule

If f is a function such that $f(x) = h(x) + g(x)$,

then $f'(x) = h'(x) + g'(x)$

That is, $\frac{d}{dx}(h+g) = \frac{d}{dx}(h) + \frac{d}{dx}(g)$.

This rule states that when the derivative is taken of terms that are added together, first the derivative of each term is taken, then the derivatives are added together. This rule works for functions with two or more terms. It also works for functions that are subtracted instead of added.

Example 4: Find the derivative of $f(x) = 4x^3 + 5x^2$.

(1)	(2)
$h = 4x^3$	$h' = 3 \cdot 4x^{3-1} = 12x^2$
$g = 5x^2$	$g' = 2 \cdot 5x^{2-1} = 10x$

<u>Step 1</u>: Separate the problem into its two terms.
<u>Step 2</u>: Use the Constant Multiple Rule to determine the derivative of each term.
<u>Step 3</u>: Add the derivatives of the terms together in the same order as the original terms.

The derivative is $f'(x) = 12x^2 + 10x$.

Example 5: Find the derivative of $f(x) = {}^{-}3x^4 + 2x^3 - 7x^2 + x - 4$.

(1)	(2)
$g = {}^{-}3x^4$	$g' = 4 \cdot \left({}^{-}3 \right) x^{4-1} = {}^{-}12x^3$
$h = 2x^3$	$h' = 3 \cdot 2x^{3-1} = 6x^2$
$i = {}^{-}7x^2$	$i' = 2 \cdot \left({}^{-}7 \right) x^{2-1} = {}^{-}14x$
$j = x$	$j' = 1 \cdot x^{1-1} = 1$
$k = {}^{-}4$	$k' = 0$

Step 1: Separate the problem into its individual terms.
Step 2: Use rules 1 - 3 to determine the derivative of each term.
Step 3: Add the derivatives of the terms together in the same order as the original terms. Since the derivative of -4 equals zero, it does not need to be included in the final answer.

The derivative is
$f'(x) = {}^{-}12x^3 + 6x^2 - 14x + 1$.

The Product Rule:

The Product Rule explains how to take the derivative of two terms that are multiplied together. First the derivative of each term is determined. Then the derivative of the first term is multiplied by the second term. Next the first term is multiplied by the derivative of the second term. Finally, the two products are added together.

The Product Rule

Let f and g be differentiable at x.
Then the product function $p = fg$ is differentiable at x, and
$p' = f'(x)\ g(x) + f(x)\ g'(x)$.

Example 6: Find the derivative of
$p(x) = \left(3x^2\right)\left(4x^5\right)$.

(1) $f = 3x^2$ $\quad g = 4x^5$
(2) $f' = 2 \cdot 3x^{2-1} = 6x$ $\quad g' = 5 \cdot 4x^{5-1} =$
(3) $p'(x) = 6x\left(4x^5\right) + 3x^2\left(20x^4\right)$
(4) $p'(x) = 24x^6 + 60x^6$
(5) $p'(x) = 84x^6$

Step 1: Separate p(x) into its individual terms.
Step 2: Use rules 1 - 3 to determine the derivative of each term.
Step 3: Now that the derivative of each term is known, the Product Rule can be completed. First, multiply the derivative of f by g. Then multiply f by the derivative of g.
Step 4: When multiplying monomials (single terms), multiply the constants in front of the variables first. Then multiply the variables that are similar by adding the exponents.

$$2x^4\left(3x^5\right) = 2 \cdot 3 \cdot x^{4+5} = 6x^9$$

Step 5: The two terms that are left are "like" terms (they have the same variable(s) and the variable(s) have the same exponents). The terms can be added together. First, add the numbers in front of the variables (24 + 60 = 84). Then write the variable. Do not change the exponent on the variable!

The derivative is $p'(x) = 84x^6$.

The Quotient Rule:

The Quotient Rule explains how to take the derivative of two terms that are divided. First, the derivative of each term is found. Then the denominator is multiplied by the derivative of the numerator. Next, the numerator is multiplied by the derivative of the denominator. Finally the second product is subtracted from the first product and that difference is divided by the denominator squared.

The Quotient Rule:

Let f and g be differentiable at x with $g(x) \neq 0$. Then the quotient $q(x) = \frac{f(x)}{g(x)}$ is differentiable at x, and

$$q'(x) = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{\left(g(x)\right)^2}$$

Example 7: Find the derivative of $q(x) = \frac{3x+4}{2x}$.

$(1)\ f(x) = 3x + 4 \qquad g(x) = 2x$

$(2)\ f'(x) = 3 \qquad g'(x) = 2$

$(3)\ q'(x) = \dfrac{2x(3) - (3x+4)(2)}{(2x)^2}$

$(4)\ q'(x) = \dfrac{6x - (6x+8)}{4x^2}$

$(5)\ q'(x) = \dfrac{6x - 6x - 8}{4x^2}$

$(6)\ q'(x) = \dfrac{-8}{4x^2}$

$(7)\ q'(x) = \dfrac{-2}{x^2}$

Step 1: Separate q(x) into its individual functions.
Step 2: Use rules 1 - 3 to determine the derivative of each term of each function.
Step 3: Now that the derivative of each function is known, the Quotient Rule can be completed. Substitute the known values of f(x), f'(x), g(x), and g'(x) into the Quotient Rule.
Step 4: Multiply 2x by 3 to get 6x. Distribute the 2 to (3x + 4) to get (6x + 8). Finally, square 2x. The (6x + 8) is placed in parentheses behind the subtraction symbol because the subtraction symbol applies to both 6x and 8.
Step 5: Distribute the subtraction symbol to the (6x + 8). This involves changing the sign of each term in the set of parentheses.
Step 6: Collect like terms. (6x - 6x = 0)
Step 7: The numerator and the denominator can both be divided by 4. Perform this

division to simplify the fraction.

The derivative is $q'(x) = \frac{-2}{x^2}$.

The Chain Rule:

The Chain Rule explains how to take the derivative of a composite function. A composite function is a function that is made up of more than one function. An example of a composite function is $f(x) = (2x + 6)^2$. One of the functions is (2x + 6) and the other function is $(x)^2$.

The Chain Rule:

If the function f(x) is differentiatable at x, and if the function g(u) is differentiatable at u(x), then the composite function $f(x) = g(u(x))$ is differentiatable at x and,

$f'(x) = g'(u(x)) \cdot u'(x)$.

To apply the Chain Rule, first determine the derivative of the outside function, then determine the derivative of the inside function. Finally, multiply the two derivatives.

Example 8:

For $f(x) = (2x^2 + 4)^2$, find f'(x).

(1) let $u(x) = 2x^2 + 4$ $\qquad u'(x) = 4$

(2) let $g(u) = u^2$ $\qquad g'(u) = 2$

(3) $g'(u(x)) = 2\left(2x^2 + 4\right)$

(4) $f'(x) = 2\left(2x^2 + 4\right) \cdot 4x$

(5) $f'(x) = \left(4x^2 + 8\right) \cdot 4x$

(6) $f'(x) = 16x^3 + 32x$

Step 1: The derivative of the inside function $\left(u(x) = 2x^2 + 4\right)$ is $u'(x) = 4x$. Step 2: The derivative of the outside function $\left(g(u) = u^2\right)$ is 2u. Step 3: Using the information we know about u(x) and g(u), we can determine g'(u(x)). We put these two together to get $g'(u(x)) = 2\left(2x^2 + 4\right)$ Step 4: Now that g'(u(x)) has been determined, the Chain Rule can be completed. Substitute the known values of g'(u(x)) and u'(x) into the Chain Rule. (For problems that are extremely difficult to simplify, this step may be considered the final answer. Since this problem is fairly easy to simplify, we will continue.)

Step 5: Distribute the 2 to both terms in the set of parentheses only. This involves multiplying both terms in the parentheses by 2.

Step 6: Distribute the 4x to the new terms in the parentheses.

The derivative is $f'(x) = 16x^3 + 32x$.

Evaluating a Derivative:

It is possible to evaluate a derivative if the value of the variable is known. In mathematics, evaluate means to substitute values in for the variables and calculate the result.

Example 9:

For $t(s) = 2s^3 + 4s^2 - 3s + 8$, evaluate $t'(s)$ at $s = 3$.

(1) $t'(s) = 6s^2 + 8s - 3$

(2) $t'(3) = 6(3)^2 + 8(3) - 3$

(3) $t'(3) = 6 \cdot 9 + 24 - 3$

(4) $t'(3) = 75$

Step 1: Since we want to evaluate t'(s), the first step is to use rules 1-4 to determine the derivative of t(s).
Step 2: Substitute 3 in place of each 's' in the derivative. Remember to use grouping symbols to group together operations which must be completed first.
Step 3: Following the order of operations, the 3 must be squared first (3 x 3 = 9). Then 8 and 3 can be multiplied.
Step 4: Multiply 6 by 9, then add that product to 24. Finally, subtract 3 from the previous sum to get 75.

The final answer is t'(3) = 75.

Vectors

A vector indicates a quantity that has both magnitude and direction. A two-dimensional vector is an ordered pair of real numbers.

A two-dimensional vector, a, can be expressed in component form as:

$a = < a_1, a_2 >$ where a_1 and a_2 are real numbers.

A three-dimensional vector, a, can be expressed in component form as:

$a = < a_1, a_2, a_3 >$ where $a_1, a_2,$ and a_3 are real n

Determining the magnitude or length of a vector:

The magnitude or length of a vector, a, is denoted by the symbol |a| or ||a||.

The length of a two-dimensional vector is found by using the following formula:

$$|a| = \sqrt{(a_1)^2 + (a_2)^2}$$

The length of a three-dimensional vector is found by using the following formula:

$$|a| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$$

Simplifying a square root:

Before we can discuss finding the length of a vector, we must briefly discuss how to simplify a radical. The following is an example of how to simplify a radical.

Example 1: Simplify the radical.

$\sqrt{300}$

(1) $\sqrt{300} = \sqrt{2 \cdot 2 \cdot 3 \cdot 5 \cdot 5}$

(2) $2 \cdot 5\sqrt{3}$

(3) $10\sqrt{3}$

Step 1: Factor the radicand (the number within the radical sign) into the product of its prime factors.
Step 2: If there are any factors that appear twice, remove them from inside the radical and write one of each pair outside the radical. Leave all other numbers inside the radical.
Step 3: Multiply the numbers that are outside of the radical. Multiply the numbers inside the radical.

Example 2: Find |a| for the vector a = <-4, 6>.

(1) $|a| = \sqrt{(-4)^2 + (6)^2}$

(2) $|a| = \sqrt{16+36}$

(3) $|a| = \sqrt{52}$

(4) $|a| = \sqrt{4 \cdot 13}$

$|a| = 2\sqrt{13}$

Step 1: Substitute the values into the formula.
Step 2: Square each number. Recall that $(-4)^2 = (-4)(-4) = 16$ Step 3: Add the numbers. Simplify the radical if possible.

Example 3: Find |b| for b = <2, 3, -5>

(1) $|b| = \sqrt{(2)^2 + (3)^2 + (-5)^2}$

(2) $|b| = \sqrt{4+9+25}$

(3) $|b| = \sqrt{38}$

Step 1: Substitute the values into the formula.
Step 2: Square each number.
Step 3: Add the numbers.
Step 4: Simplify the radical, if possible.

Vector addition and subtraction:

For two-dimensional vectors:
If $a = \langle a_1, a_2 \rangle$ and $b = \langle b_1, b_2 \rangle$, then $a + b$ is expressed as $\langle a_1 + b_1, a_2 + b_2 \rangle$ and a - b is expressed as $\langle a_1 - b_1, a_2 - b_2 \rangle$.

For three-dimensional vectors:
If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, then $a + b$ is expressed as $\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$ and a - b is expressed as $\langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$.

Example 4: If a = < 2, 3> and b = <-3, -7> find a + b.

(1) a + b = <2 + -3, 3 + -7>
(2) a + b = <-1, -4>

Step 1: Add a_1 to b_1 and a_2 to b_2. Step 2: Add the values.

Example 5: If a = < 2, 3> and b = <-3, -7> find a - b.

(1) a - b = <2 - -3, 3 - -7>
(2) a - b = <2 + 3, 3 + 7>
(3) a - b = < 5, 10>

Step 1: Subtract b_1 from a_1 and b_2 from a_2. Step 2: Remember that subtracting a negative is the same as adding a positive.
Step 3: Subtract the values.

Multiplication of a vector by a scalar:

Two dimensional vectors:

If 's' is a scalar and $a = \langle a_1, a_2 \rangle$, then $(s)(a) =$

Three-dimensional vectors:

If 's' is a scalar and $a = \langle a_1, a_2, a_3 \rangle$, then $(s)(a)$

Example 6: If a = < 2, 3> and b = <-3, -7> find 3a.

(1) 3a = 3<2, 3>
(2) 3a = <3(2), 3(3)>
(3) 3a = <6, 9>

Step 1: Substitute the values into the formula.
Step 2: Multiply each entry of the vector by the scalar.
Step 3: Simplify.

Example 7: If a = < 2, 3> and b = <-3, -7> find 4a + 2b.

(1) 4a + 2b = 4<2, 3> + 2<-3, -7>

(2) 4a + 2b = <4(2), 4(3)> + <2(-3), 2(-7)>

(3) 4a + 2b = <8, 12> + <-6, -14>

(4) 4a + 2b = <8 + -6, 12 + -14>

(5) 4a + 2b = <2, -2>

Step 1: Substitute the values into the formula.
Step 2: Multiply each entry of the vector by the scalar.
Step 3: Simplify.
Step 4: Add the corresponding entries of the vectors.
Step 5: Simplify.

Standard basis (or unit) vectors:

The following vectors each have length of one:

i = <1, 0> and **j = <0, 1>** since:

$$|i| = \sqrt{(1)^2 + (0)^2} = \sqrt{1+0} = \sqrt{1} = 1$$

and

$$|j| = \sqrt{(0)^2 + (1)^2} = \sqrt{0+1} = \sqrt{1} = 1$$

These vectors are called the standard basis (or unit) vectors in two dimensions. Similarly, **i = < 1, 0, 0>**, **j = < 0, 1, 0>**, **k = <0, 0, 1>** are the standard basis vectors in three dimensions. Each vector has a length of one.

Any two-dimensional vector can be written in terms of i and j.

If $a = \langle a_1, a_2 \rangle$, then $a = a_1 i + a_2 j$.

Any three-dimensional vector can be written in terms of i, j, and k.

If $a = \langle a_1, a_2, a_3 \rangle$, then $a = a_1 i + a_2 j + a_3 k$.

For example, in two-dimensions, if b = <-5, 4>, then b = -5i + 4j. In three-dimensions, if a = < 6, 1, -3> then a = 6i + 1j - 3k. You can also make -2i + 3j - 4k = <-2, 3, -4> and 6i + 3j = <6, 3>.

Example 8: Find |a| when a = 3i + 2j.

(1) $|a| = \sqrt{(3)^2 + (2)^2}$

(2) $|a| = \sqrt{9+4}$

(3) $|a| = \sqrt{13}$

Step 1: Substitute the values into the formula to find the length of a vector.
Step 2: Square each number.
Step 3: Add the squares of the numbers. Simplify the radical, if possible.

Example 9: Find a + b when a = 3i + 2j and b = -4i - 5j.

(1) a + b = (3i + 2j) + (-4i - 5j)

(2) a + b = (3i - 4i) + (2j - 5j)

(3) a + b = -1i - 3j

Step 1: Substitute the values into the formula for adding vectors.
Step 2: Collect like terms.
Step 3: Add or subtract the like terms.

Example 10: Find 2a + 4b when a = 3i + 2j and b = -4i - 5j.

(1) 2a + 4b = 2(3i + 2j) + 4(-4i - 5j)

(2) 2a + 4b = (6i + 4j) + (-16i - 20j)

(3) 2a + 4b = (6i - 16i) + (4j - 20j)

(4) 2a + 4b =-10i - 16j

Step 1: Substitute the vectors into the formula 2a + 4b.
Step 2: Apply the distributive property. This involves multiplying each term in each set of parentheses by the scalar that is in front of it.
Step 3: Collect like terms.
Step 4: Add or subtract like terms.

Dot product:

The <u>dot product</u> is another vector operation that determines a scalar value (which is a real number, not a vector).

For two-dimensional vectors:

If $a = \langle a_1, a_2 \rangle$ and $b = \langle b_1, b_2 \rangle$, then the dot product of a and b is $a \cdot b = a_1b_1 + a_2b_2$

For three-dimensional vectors:

If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, then the dot product of a and b is
$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$.

Example 11: Find the dot product of a = < -1, 2> and b = <4, 3>.

(1) $a \cdot b = -1(4) + 2(3)$

(2) $a \cdot b = -4 + 6$

(3) $a \cdot b = 2$

<u>Step 1</u>: Substitute the values into the formula for the dot product of a two-dimensional vector.
<u>Step 2</u>: Multiply the numbers.
<u>Step 3</u>: Add the numbers.

Example 12: Find the dot product of a = 4i + 2j - 3k and b = 2i - 3j + k

(1) $a \cdot b = 4(2) + 2(-3) + -3(1)$

(2) $a \cdot b = 8 + -6 + -3$

(3) $a \cdot b = -1$

<u>Step 1</u>: Substitute the values into the formula for the dot product of a three-dimensional vector.
<u>Step 2</u>: Multiply the numbers.
<u>Step 3</u>: Add the numbers.

Cross product:

The cross product is an operation on vectors that is denoted by an 'x' and results in another vector.

If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, then the
$a \times b = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$.

Example 13: Find the cross product of a = < -2, 3, 4> and b = <2, 4, 0>

(1) a x b = < 3(0) -4(4), 4(2) - -2(0), -2(4) - 3(2)>
(2) a x b = <0 - 16, 8 + 0, -8 - 6>
(3) a x b = <-16, 8, -14>

<u>Step 1</u>: Substitute the values into the formula for the cross product of vectors.
<u>Step 2</u>: Multiply the numbers.
<u>Step 3</u>: Add or subtract the numbers.

Example 14: Find the cross product of a = < 1, 2, -3> and b = <-3, 1, 6>

(1) a x b = < 2(6) --3(1), -3(-3) - 1(6), 1(1) - 2(-3)>
(2) a x b = <12 + 3, 9 - 6, 1 + 6>
(3) a x b = <15, 3, 7>

<u>Step 1</u>: Substitute the values into the formula for the cross product of vectors.
<u>Step 2</u>: Multiply the numbers.
<u>Step 3</u>: Add the numbers.

Limits

The <u>limit</u> of a function is the value approached by a function as the independent variable approaches some value. This can be defined mathematically as:

$$\lim_{x \to a} f(x) = L$$

The limit of the function f(x) as x approaches a is a constant L. This definition presumes that f(x) becomes very close to L as x approaches a from either side.

The limit properties are:

$$\text{(a) } \lim_{x\to a}\left(f(x)+g(x)\right)=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$$

$$\text{(b) } \lim_{x\to a}\left(f(x)-g(x)\right)=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$$

$$\text{(c) } \lim_{x\to a}\left(cf(x)\right)=c\lim_{x\to a} f(x), \text{where c is a constant}$$

$$\text{(d) } \lim_{x\to a}\left(f(x)\cdot g(x)\right)=\lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$$

$$\text{(e) } \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \text{if } \lim_{x\to a} g(x)\neq 0$$

A description of each property follows.

(a) The limit of a sum equals the sum of the limits.
(b) The limit of a difference equals the difference between the limits.
(c) The limit of a constant times a function equals the constant times the limit of the function.
(d) The limit of a product equals the product of the limits.
(e) The limit of a quotient equals the quotient of the limits.

Example 1: Evaluate the limit.

$$\lim_{x\to 1}(3x^2+2x+4)$$

$$(1)\ \lim_{x\to 1} 3x^2+\lim_{x\to 1} 2x+\lim_{x\to 1} 4$$

$$(2)\ 3(1)^2+2(1)+4$$

$$(3)\ 3+2+4$$

$$(4)\ 9$$

Step 1: Use the addition property to split the expression into individual limit terms.
Step 2: Compute the limit by substituting (1) in for x.
Step 3: Calculate by squaring the 1 (1 x 1 = 1) and then multiplying.
Step 4: Add the values.

Example 2: Evaluate the limit.

$$\lim_{x\to {}^{-}5}\frac{x^2+8x+15}{x+5}$$

$$(1)\ \lim_{x\to {}^{-}5}\frac{(x+5)(x+3)}{(x+5)}$$

$$(2)\ \lim_{x\to {}^{-}5} x+3$$

$$(3)\ {}^{-}5+3$$

$$(4)\ {}^{-}2$$

Step 1: Factor the numerator and denominator. In this case, the denominator is already factored. The numerator factors to (x + 5)(x + 3).
Step 2: Divide out the common factor (x + 5).
Step 3: Substitute (-5) in for x.
Step 4: Add the values.

NOTE: This limit cannot be evaluated until Step 2 is completed. If -5 is substituted into the original function, the denominator becomes x + 5 = -5 + 5 = 0. A fraction with a denominator of zero is undefined. By factoring and dividing out any common factors before substituting -5 in place of x, the possibility of an undefined fraction is eliminated.

Example 3: Evaluate the limit.

$$\lim_{s\to 0}\frac{8s^4+5s^2-3s}{s}$$

$$(1)\ \lim_{s\to 0}\frac{\cancel{s}\left(8s^3+5s-3\right)}{\cancel{s}}$$

$$(2)\ \lim_{s\to 0} 8s^3+5s-3$$

$$(3)\ 8(0)^3+5(0)-3$$

$$(4)\ -3$$

Step 1: Factor the numerator and denominator. In this case, the denominator is already factored.
Step 2: Divide out the common factor, s.
Step 3: Substitute (0) in for s.
Step 4: Calculate by simplifying all numbers with exponents and then multiplying.
Step 5: Add the values.

Example 4: Evaluate the limit.

$$\lim_{y\to 0}\frac{(y+3)^2-9}{y}$$

$$(1)\ \lim_{y\to 0}\frac{y^2+6y+9-9}{y}$$

$$(2)\ \lim_{y\to 0}\frac{y^2+6y}{y}$$

$$(3)\ \lim_{y\to 0}\frac{y(y+6)}{y}$$

$$(4)\ \lim_{y\to 0}(y+6)$$

$$(5)\ 0+6$$

$$(6)\ 6$$

Step 1: Multiply the binomial $(y+3)^2=(y+3)(y+3)=y^2+6y+9$. Step 2: Combine like terms, 9 and -9.
Step 3: Factor the numerator and denominator. In this case, the denominator is already factored.
Step 4: Divide out the common factor, y.
Step 5: Compute the limit by substituting (0) for y.
Step 6: Add the values.

Example 5:

Find the $\lim_{x\to 3} f(x)$, where $f(x)=\begin{cases} x^2 \text{ if } x=4 \\ x+4 \text{ if } x\neq 4\end{cases}$

(1) f(x) = x + 4
(2) f(3) = 3 + 4
(3) f(3) = 7

Step 1: We want to find the limit as x approaches 3, so x = 3. Since x = 3, $x \neq 4$. Therefore, we need to evaluate the limit at f(x) = x + 4.
Step 2: Compute the limit by substituting 3 in for x.
Step 3: Add the values.